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LAGUERRE POLYNOMIALS VIA INTEGER PARTITIONS AND EULER'S TOTIENT FUNCTION

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Abstract. We employ the Jameson-Schneider and Fine theorems to obtain a process based in integer partitions to construct the Laguerre polynomials.

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1. Introduction

Here we consider the functions:

$$\psi_j(q) := (1-q)^{s(j)} = \sum_{n=0}^{\infty} C_j(n) q^n, \tag{1}$$

for an arbitrary sequence $\{s(n)\}$, and we accept that:

$$\eta(q) := \Pi_{j=1}^{\infty}\, \psi_j(q^j) = \Pi_{j=1}^{\infty}\, (1-q)^{s(q)} = \sum_{n=0}^{\infty} R(n)\, q^n, \quad R(0) = 1; \tag{2}$$

on the other hand, Jameson-Schneider [1, 2] obtained the following result:

$$q\,\frac{d}{dq}\ln \eta \; = -\sum_{n=1}^{\infty} \Big(\sum_{d/n} s(d)\; d \Big)\, q^n. \tag{3}$$

Besides, we know the Fine's theorem [3, 4, 5]:

$$\eta = \sum_{n=0}^{\infty} \Big(\sum_{\lambda \,\vdash\, n} C_1(k_1)\; C_2(k_2)\; ...\; C_n(k_n) \Big)\, q^n, \tag{4}$$

where $\lambda \vdash n$ means all partitions of n, and k_r is the multiplicity of r in a given partition; so, (2) and (4) imply the relationship:

$$R(n) = \sum_{\lambda \,\vdash\, n} C_1(k_1)\; C_2(k_2)\; ...\; C_n(k_n). \tag{5}$$

In Sec. 2 we employ the expressions (1),..., (5) for the case $s(j) = \varphi(j)x/j$ in terms of Euler's totient function [6, 7, 8, 9] to show that the partitions of an integer allow construct the Laguerre polynomials [10, 11, 12, 13, 14].

2. Laguerre polynomials and integer partitions

From (2) and (3) is immediate the following recurrence relation:

$$nR(n) = \sum_{j=1}^{n} h(j)\ R(n-j), \qquad h(j) = -\sum_{d/j} s(d)\ d, \qquad h(0) = 0, \tag{6}$$

whose solution is given by [15]:

$$R(n) = \frac{1}{n!}\ B_n\Big(0!\ h(1), 1!\ h(2), 2!\ h(3), ..., (n-1)!\ h(n)\Big), \tag{7}$$

in terms of the complete Bell polynomials [15, 16], with the corresponding inversion [17] for $n \geq 1$:
(8)

$$(n-1)!\ \sum_{d/n} s(d)\ d = \sum_{k=1}^{n} (-1)^k (k-1)!\ B_{n,k}\Big(1!\ R(1), 2!\ R(2), ..., (n-k+1)!\ R(n-k+1)\Big),$$

with the presence of the partial Bell polynomials [17, 18]; besides, the coefficients in the expansion (1) can be calculated with the relation:

$$C_j(n) = \frac{(-1)^n}{n!}\ \Pi_{t=0}^{n-1}\Big(s(j)-t\Big) = \frac{1}{n} C_j(n-1)\Big(C_j(1)+n-1\Big),\ n \geq 1,\ C_j(0) = 1. \tag{9}$$

Now we shall apply our expressions in the case $s(m) = \varphi(m)x/m$ involving the Euler's totient function [6, 7, 8, 9], therefore $\sum_{d/n} s(d)\ d = x \sum_{d/n} \varphi(d) = nx$ [19], with the property [20, 21]:
(10)

$$\Pi_{j=1}^{\infty}(1-q^j)^{\varphi(j)x/j} = e^{-\frac{qx}{1-q}} = \sum_{k=0}^{\infty} R(k) q^k, \ \text{ therefore } \ R(n) = \sum_{t=0}^{n} \binom{n-1}{t-1} \frac{(-x)^t}{t!},$$

thus (8) implies the identity:

$$n!\ x = \sum_{k=1}^{n} (-1)^k (k-1)!\ B_{n,k}\Big(1!\ R(1), 2!\ R(2), ..., (n-k+1)!\ R(n-k+1)\Big), \tag{11}$$

such that:

$$R(1) = -x, \qquad R(2) = -x + \frac{x^2}{2}, \qquad R(3) = -x + x^2 - \frac{x^3}{6}, ... \tag{12}$$

However, here we search a process based in integer partitions to construct the Laguerre polynomials whose generating function is given by [10, 11, 12, 13, 14]:

$$\frac{1}{1-q}\ e^{-qx/(1-q)} = \sum_{k=0}^{\infty} L_k(x)\ q^k, \tag{13}$$

and its comparison with (10) implies:
(14)

$$\sum_{k=0}^{\infty} R(k) q^k = (1-q) \sum_{k=0}^{\infty} L_k(x) q^k, \ \text{ therefore } \ R(n) = L_n(x) - L_{n-1}(x) = L_n^{-1}(x), \ \ n \geq 1,$$

then from (5) and (14):

$$L_n(x) = L_{n-1}(x) + \sum_{\lambda \,\vdash\, n} C_1(k_1)\ C_2(k_2)\ ...\ C_n(k_n), \ \ L_0(x) = 1, \tag{15}$$

which is a recurrence relation in terms of integer partitions, and from (9):

$$C_j(1) = -\frac{\varphi(j)x}{j}, \quad C_j(n) = \frac{1}{n}\, C_j(n-1)\left(n-1-\frac{\varphi(j)x}{j}\right), \;\; n \geq 1, \tag{16}$$

with the important participation of Euler's totient function; thus, for example:

$$\begin{aligned} &L_1(x) = L_0(x)+C_1(1) = 1-x, \quad L_2(x) = L_1(x)+C_1(2)+C_2(1) = \frac{1}{2}\Big(2-4x+x^2\Big),\\ &L_3(x) = L_2(x)+C_1(3)+C_1(1)C_2(1)+C_3(1) = \frac{1}{6}\Big(6-18x+9x^2-x^3\Big), ..., \end{aligned} \tag{17}$$

therefore, (15) gives a method based in integer partitions to obtain the Laguerre polynomials.

Remark 1.- The relations (7) and (14) imply the identity:

$$L_n^{-1}(x) = \frac{1}{n!}\, B_n\Big(-1!\,x, -2!\,x, -3!\,x, ..., -n!\,x\Big) = -\frac{x}{n} L_{n-1}^{1}(x). \tag{18}$$

Remark 2.- The expression (10) can be written in terms of the Lah numbers [22, 23, 24, 25, 26], in fact:

$$R(n) = \frac{1}{n!}\, \sum_{k=0}^{n} (-1)^k L_n^{[k]}\, x^k, \qquad L_n^{[k]} = \binom{n-1}{k-1}\frac{n!}{k!}, \tag{19}$$

which accepts the inversion [27]:

$$x^n = \sum_{k=0}^{n} (-1)^k\, L_n^{[k]}\, k!\, R(k) = \sum_{k=0}^{n} (-1)^k L_n^{[k]}\, k! \sum_{\lambda \vdash k} C_1(k_1)\, ...\, C_n(k_n), \tag{20}$$

with the $C_j(m)$ given by (16).

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